RESOURCES, INSTITUTIONS AND TECHNOLOGIES: GAME MODELING OF DUAL RELATIONS*

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Abstract
A new approach is proposed revealing duality relations between a physical side of economy (resources and technologies) and its institutional side (distributional relations between social groups). Production function is modeled not as a primal object but rather as a secondary one defined in a dual way by the institutional side. Differential games of bargaining are proposed to model a behavior of workers and capital-owners in process of prices or weights formation. These games result, correspondingly, in a price curve and in a weight curve – structures dual to a production function. Ultimately, under constant bargaining powers of the participants, the Cobb-Douglas production function is generated.
Key words: *Production function, Production factors, Choice of technology, Bargaining, Differential games, Duality*

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1. Introduction

Can institutions be a primal reason of using definite technologies in the economy? The paper studies this question in relation to the problem of micro foundations of production function. Acceptance of concrete types of production functions in economics, such as Cobb-Douglas and CES, was rather occasional and since now not enough attempts have been made to explain and justify the widely used types of production function – see Acemoglu, 2003, Jones, 2005, Lagos, 2007, Nakamura, 2009, Matveenko, 2010, Dupuy, 2012. In the paper models resulting in the Cobb-Douglas production function are constructed on base of dual relations between production and institutional sides of the economy, by use of differential games of bargaining.

Duality is being considered both in general economics texts (e.g. Jehle and Reny, 2001) and in more specialized ones (e.g. Cornes, 1992). Diewert (1982) reviews numerous professional publications enlightening on the role of economic duality. An interesting historical note on duality in the production theory is provided by Jorgenson and Lau (1974). Knowledge of any one of a pair of dual objects is enough for restoring the other one. The spreading of duality in economics is connected with the mathematical fact that closed convex sets (which are often found in economic systems) can be described in two ways: by enumerating their elements (a primal description) and by enumerating closed subspaces containing the set (a dual description).

One way to exploit the dual relations is to use them as a basis for studying relations between a physical side and an institutional side of production process. These two sides of the economic system are in some duality relations; and it is a widespread view in the literature that institutions play equally important role in production as physical resources and physical technologies do. Jorgenson and Lau (1974) study an equivalence of technological and behavioral approaches as a basis of the production theory. Stern (1991), Hall and Jones (1999), Acemoglu et al. (2005), Acemoglu and Robinson (2010) consider social infrastructure and other institutions as determinants of economic growth and show that differences in incomes between countries are in a con-

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siderable measure explained by differences in social infrastructure. Nelson and Sampat (2001) and Nelson (2008) relate institutions to social technologies which are used in production symmetrically to physical technologies. Papandreou (2003) argues that “Though it is often difficult to distinguish institutional and physical constraints impinging on production and consumption sets, it is important to do so, as it provides a starting point for what can and cannot be controlled by human agency”. However, an idea of duality of resources and institutions, physical and social technologies is still at a stage of formation.

In the paper a simple differential game of price bargaining is introduced as a benchmark and then is modified to a differential game of formation of prices of capital and labor and to a differential game of formation of weights (assessments) of the factor-owners. Each of these three differential games exploits one or another of duality relations existing in economies. The price curve and the wage curve are dual object in relation with the factor curve. Their formation is modeled by use of differential bargaining games. Ultimately, under constant bargaining powers of the participants, the Cobb-Douglas production function is generated.

The paper is organized as follows. Section 2 describes basic duality relation to be used in the paper. In Section 3 a benchmark differential game of price bargaining is introduced. In Section 4 a differential game of factor price formation is considered resulting in a price curve which leads to the Cobb-Douglas production function. In Section 5 a differential game of weight curve formation is studied and its relation to the production function is shown. Section 6 concludes.

2. Some dual relations

One of the duality relations used in the paper is usually represented as the duality between the production function, $Y = F(K, L)$, and the cost function, $C(p_K, p_L, Y)$; here Y is output, K is capital, L is labor, p_K and p_L are the factor prices. The first of these two functions shows the maximal output in dependence on the production factors while the second one shows the minimal cost given factor prices and an output. We propose a model of a process of the factor prices formation which is based on the institutional side of production and is independent on the physical side. To do this we use a duality correspondence between the production function and a set of the factor prices.

The following representation of the production function by use of the Euler theorem is often used in the economic growth theory:

$$F(K, L) = \frac{\partial F}{\partial K} K + \frac{\partial F}{\partial L} L = p(x)x\,,$$

where $x = (K, L)'$ is the vector of production factors and $p(x) = \left(\frac{\partial F}{\partial K}, \frac{\partial F}{\partial L} \right)$ is the corresponding price vector (the vector of marginal productivities). There exists a set Π of the price vectors corresponding the production function, such that the Euler theorem can be written in an ‘extremal’ form:

$$F(K, L) = \min_{p \in \Pi} px\,, \quad (1)$$

which means that the production function represents a result of a choice of a price vector from the set Π. The set Π is referred to as a *price curve*.

Let $M = \{x : F(x) = 1\}$ be the unit level line of the production function *F*; it will be referred as a *factor curve*. A conjugate problem for (1) is the problem of a choice of a bundle of production factors $x = (K, L)$ from the factor curve to provide a unit output with minimal cost:

$$F^*(p) = \min_{x \in M} px\,.$$

Rubinov (Rubinov and Glover, 1998, Rubinov, 2000) was the first to study new types of duality using, instead of the usual inner product, $px = \sum_{i=1,\dots,n} p_i x_i$, its analogues such as the Leontief function, $\min_{i=1,\dots,n} l_i x_i$. Notably, the latter is similar to the inner product but uses an 'idempotent summation' operation $\oplus = \min$. Matveenko (1997, 2010) and Jones (2005) found for neoclassical production functions a representation which reminds (1) but uses the Leontief function:

$$F(K,L) = \max_{\lambda\in\Psi} \min\{l_K K,\, l_L L\}.$$

Here Ψ is a *technological menu* which corresponds the production function *F*.

In the present paper both the usual and the generalized types of duality are used. The ordinary duality allows us to construct a microfoundation of production functions on base of the price curve. The generalized duality introduced by Rubinov and Glover, 1998, Rubinov, 2000 and Matveenko, 1997 is used to make the microfoundation more precise by specifying in what way the income distribution and a corresponding choice of technology can depend on formation of a set of moral-ethical assessments (weights) by the owners of the resources in dependence on their bargaining powers; this set will be referred as a *wage curve*.

3. Benchmark differential game of price bargaining

The term *bargain* relates both to a process of bargaining and to a result of this process. Both sides of bargaining are being studied in the bargaining theory – a special chapter of the game theory. However, traditionally, the bargaining theory deals more with results of bargains rather than with processes of bargaining. Nash (1950) proposed a system of axioms leading to a so called symmetric Nash bargaining solution; later an asymmetric solution was found and axiomatized[2]. For the reviews of the axiomatic approach in the bargaining see Roth, 1979, Thomson and Lendsberg, 1989, Thomson, 1994, Serrano, 2008.

The models of processes of bargaining are usually based on assumptions concerning economic benefits gained by participants under one or other running of the process of bargaining (see Muthoo, 1999). For example, a participant can bear some costs connected with the duration of the bargaining process. In practice, however, in many cases the course of a bargaining process depends in much not on expectations of economic benefits by participants but on their skills to bargain[3]. These skills can be associated with *bargaining powers* of the participants. The notion of bargaining power is often used in game theory, though, different authors put different sense into this notion[4].

In this Section we propose a simple differential game as a model of a bargaining process. In different versions of the game the bargaining powers of the players are either given exogenously or are defined endogenously in the game itself.

In the benchmark example of bargaining (Muthoo, 1999) an object is on sale (e.g. a house). A seller (player S) wishes to sell the house for a price exceeding $\overline{p}_S^0$ (the latter is the minimal price acceptable for player S). A buyer (player B) is ready to purchase the house for a price not exceeding $\overline{p}_B^0$ (the maximal acceptable price for player B). Here $\overline{p}_B^0 > \overline{p}_S^0$, what ensures the possibility

[2] An asymmetric Nash bargaining solution satisfies axioms of Pareto optimality, independence on irrelevant alternatives, and independence on a linear transformation.

[3] In fiction we can find descriptions of bargaining processes where bargaining powers of players are of the first importance and these powers are not related directly to any economic benefits as in the following example. *"I went to look after a piece of old brocade in Wardour Street and had to bargain for hours for it. Nowadays people know the price of everything and the value of nothing".* Oscar Wilde. The Picture of Dorian Gray.

[4] Ways of behavior of bargainers are being studied in a so called *tactical approach* initiated by Schelling (1956). In the present paper the bargaining models are very schematic; but it can be expected that detailed tactical models applied in a similar would provide interesting results.

of the bargain. The seller starts from a start price, $p_S(0) > \overline{p}_S^0$, and then decreases her price, while the buyer simultaneously starts from a price $p_B(0) < \overline{p}_B^0$ and then increases her price. It is assumed, naturally, that $p_B(0) < p_S(0)$. A price trajectory $(p_B(t),\ p_S(t))$ formed in continuous time stops at a moment *T* when $p_B(T) = p_S(T)$. It follows that $p_B(t) < p_S(t)$ for $t \in [0,\ T)$. The selling price will be referred as p^*.

A surplus of the selling price over (under) the minimal (maximal) admissible price of a player can be considered as the player's utility:

$$u_S = p^* - \overline{p}_0^S,\ u_B = \overline{p}_0^B - p^*. \quad (2)$$

A set Ω of all possible pairs of utilities on plane (u_B, u_S) is

$$\{(u_B, u_S) :\ u_B + u_S = \overline{p}_0^B - \overline{p}_0^S,\ u_B, u_S \geq 0\}.$$

A simplest model of price bargaining appears under an assumption that each player $i = B,\ S$ changes her price with a constant velocity equal to the bargaining power of her opponent, b_j:

$$p_i(t) = p_i(0) + \dot{p}_i t, \quad i = B, S,$$
$$\dot{p}_S = -b_B,\ \dot{p}_B = b_S.$$

A strong opponent forces the player to change her price faster[5]. Hence,

$$p_S(t) = p_S(0) - b_B t,\ p_B(t) = p_B(0) + b_S t.$$

The game stops at the moment *T* which is found from equation:

$$p_0^S - b_B T = p_B^0 + b_S T,$$

i.e. at the moment

$$T = \frac{p_S(0) - p_B(0)}{b_S + b_B}$$

when the selling price is:

$$p^* = p_S(T) = p_B(T) = \frac{b_S}{b_S + b_B} p_S(0) + \frac{b_B}{b_S + b_B} p_B(0). \quad (3)$$

So, the selling price is the convex combination of the start prices proposed by the players summed with weights equal to their relative bargaining powers.

Now let each player *i* know the minimal (maximal) price $\overline{p}_j^0$ accessible for the opponent and establish this as her start price: $p_S(0) = \overline{p}_B^0,\ p_B(0) = \overline{p}_S^0$. Then the play stops at the moment:

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[5] Similar results would be received if it is assumed that the velocity of changing the price by player *i* is inversely proportional to her own bargaining power: a high bargaining power means that the player agrees only to small abatement in bargaining.

$$T = \frac{\overline{p}_B^0 - \overline{p}_S^0}{b_S + b_B}$$

with the selling price

$$p^* = \frac{b_S}{b_S + b_B}\overline{p}_B^0 + \frac{b_B}{b_S + b_B}\overline{p}_S^0$$

And with the utilities of the players equal to

$$u_S = \frac{b_S}{b_S + b_B}(\overline{p}_B^0 - \overline{p}_S^0), \ u_B = \frac{b_B}{b_S + b_B}(\overline{p}_B^0 - \overline{p}_S^0). \quad (4)$$

PROPOSITION 1. *Price* p^* *corresponds the asymmetric Nash bargaining solution of the bargaining problem under utilities* (2) *and bargaining powers* b_S, b_B.

Proof. The asymmetric Nash bargaining solution is here a solution of the problem of maximization of the function $u_B^{b_B} u_S^{b_S}$ on the set Ω. The first order optimality condition and the constraint form the system:

$$\begin{cases} b_B u_S = b_S u_B \\ u_B + u_S = \overline{p}_B^0 - \overline{p}_S^0 \end{cases},$$

from which the asymmetric Nash bargaining solution is found:

$$u_i = \frac{b_i}{b_S + b_B}(\overline{p}_B^0 - \overline{p}_S^0), \quad i = B, S,$$

which coincides with (4). Q.E.D.

The case when the players change prices under constant growth rates (rather than constant velocities) equal to their bargaining powers is similar. Since the growth rate g_i of price $p_i(t)$ $(i = B,\ S)$ is the velocity of changing the logarithm of the price, $\ln p_i(t)$, an equation similar to (3) is fulfilled, and the bargaining stops under a price p^{**} the logarithm of which is equal to the convex combination of the logarithms of the start prices with weights equal to the relative bargaining powers of the players:

$$\ln p^{**} = \frac{b_S}{b_S + b_B}\ln p_S(0) + \frac{b_B}{b_S + b_B}\ln p_B(0).$$

In a more complex case the velocity of changing price by a player depends on the actions of her opponent. If the seller decreases her price slowly then the buyer also increases her price slowly because she does not want the game to stop on a too high price. Similarly, if the buyer increases her price slowly then the seller decreases her price slowly. The following system of equations can serve as a model:

$$v_S = -\dot{p}_S = f_S(v_B), \ \ v_B = \dot{p}_B = f_B(v_S), \quad (5)$$

where it is natural to assume that

$$f_S(0) = f_B(0) = 0$$

– a player changes her price only if the opponent changes hers; moreover,

$$f_S{}'(0) = f_B{}'(0) = 0,$$

and the functions $f_S(.) = f_B(.)$ are increasing and strictly convex: velocity of any player increases when the opponent changes her price faster.

Under the present conditions there exists a unique equilibrium pair of velocities (v_S, v_B) satisfying the system of equations (5). This pair is a Nash equilibrium: no one player alone wishes to change her velocity of price change.

EXAMPLE. Let

$$v_S = a_S v_B^2, \; v_B = a_B v_S^2,$$

then

$$v_S = a_S^{-1/3} a_B^{=2/3}, \; v_B = a_S^{-2/3} a_B^{-1/3}.$$

The game stops at the moment

$$T = \frac{\overline{p}_B^0 - \overline{p}_S^0}{a_S^{-1/3} a_B^{-2/3} + a_B^{-2/3} a_S^{-1/3}}$$

under the selling price

$$p^* = \frac{a_S^{-1/3}}{a_S^{-1/3} + a_B^{-1/3}} \overline{p}_S^0 + \frac{a_B^{-1/3}}{a_B^{-1/3} + a_S^{-1/3}} \overline{p}_B^0.$$

The coefficients $a_S^{-1/3}$, $a_B^{-1/3}$ can be interpreted as the players' bargaining powers: the smaller the bargaining power is the stronger the player's reaction to her opponent's price change is.

Let the growth rates of price change, $g_i = \frac{\dot{p}_i}{p_i}, \; i = B, S$, be constant. The bargaining power of player *i* can be defined as the value inverse to $|g_i|$:

$$b_B = \frac{1}{g_B}, \; b_S = -\frac{1}{g_S}.$$

Then

$$\frac{g_B}{g_S} = -\frac{b_S}{b_B}, \quad (6)$$

i.e.

$$\frac{dp_B}{dp_S} \frac{p_S}{p_B} = -\frac{b_B}{b_S} = const,$$

The game interpretation of this differential equation is the following. Each player *i* chooses a control g_i, and the controls are connected by the relation:

$$g_B \geq |g_S| \frac{b_S}{b_B}, \quad (7)$$

which means that in the bargaining process the faster the seller decreases her price the faster the buyer increases hers. Moreover, a high bargaining power of the buyer relaxes this constraint (this means a lower degree of reaction to the opponent's actions), and a high bargaining power of the seller reinforces the constraint.

At the same time the seller is limited by the opposite constraint:

$$|g_S| \geq g_B \frac{b_B}{b_S}, \quad (8)$$

which means that the faster the buyer increases her price the faster the seller decreases hers. An increased bargaining power of the buyer forces the seller to diminish her price faster, and an increased own bargaining power allows the seller to diminish her price slower.

Simultaneous fulfillment of inequalities (7) and (8) implies the Equation (6).

4. Bargaining for production factor prices and corresponding choice of technologies

In the just described benchmark differential game the players change their proposals concerning the same price. Now we turn to differential games in which the interests of the players relate to different prices. At each moment of time one of the players *attacks*, another one *defends*. Only the attacker is satisfied by the direction of her price change while the defender hinders changes in her price.

In the present Section the following pair of dual objects will be under consideration:

(i) a neoclassical production function $F(K, L)$ which is characterized by its factor curve: $M = \{(K, L): F(K, L) = 1\}$ i.e. the set of bundles of resources allowing the unit output, and

(ii) the price curve $\Pi = \{(p_K, p_L)\}$ i.e. the set of such bundles of prices under which the unit output under unit costs is possible.

4.1 Usual causality

Given production function $F(K, L)$ the price curve Π can be found from the following system of equations:

$$F(K, L) = 1, \qquad (9)$$

$$p_K K + p_L L = 1, \quad (10)$$

$$\frac{\partial F / \partial K}{\partial F / \partial L} = \frac{p_K}{p_L}, \quad (11)$$

Equations (9) and (10) are conditions of the unit output under unit costs. Equation (11) is a condition of efficiency of production; it can be interpreted as a condition of maximization of output under given costs.

The system (9)-(11) establishes a one-to-one correspondence between points of the factor curve, *M*, and points of the price curve, Π. Indeed, by the Euler theorem, the Equation (9) can be written as

$$\frac{\partial F}{\partial K} K + \frac{\partial F}{\partial L} L = 1, \quad (12)$$

then the Equations (10)-(12) imply:

$$\frac{\partial F}{\partial K} = p_K, \ \frac{\partial F}{\partial L} = p_L. \quad (13)$$

In particular, for the Cobb-Douglas production function, $F(K,L) = AK^{\alpha} L^{1-\alpha}$, the system (13) takes the form:

$$\alpha A K^{\alpha-1} L^{1-\alpha} = p_K,$$
$$(1-\alpha) A K^{\alpha} L^{-\alpha} = p_L.$$

Excluding the ratio K / L from these two equations we find the price curve Π:

$$\left(\frac{p_K}{\alpha A}\right)^{\frac{1}{\alpha-1}} = \left(\frac{p_L}{(1-\alpha)A}\right)^{\frac{1}{\alpha}}.$$

After raising both sizes of the equation to power $\alpha(\alpha-1)$, the price curve takes the form:

$$Bp_K^{\alpha} p_L^{1-\alpha} = 1,$$

where $B = A^{-1}\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}$.

For the CES function $F(K,L) = A(\alpha(A_K K)^p + (1-\alpha)(A_L L)^p)^{1/p}$ where $p \in (-\infty, 0) \cup (0,1)$ the system (13) has the form:

$$\begin{cases} \alpha A_K^p \left(\alpha A_K^p + (1-\alpha)K^{-p}(A_L L)^p\right)^{\frac{1}{p}-1} = p_K \\ (1-\alpha)A_L^p \left(\alpha (A_K K)^p L^{-p} + (1-\alpha)A_L^p\right)^{\frac{1}{p}-1} = p_L \end{cases}.$$

Excluding $K^{-p}L^p$ from these equations we receive:

$$\frac{\left(\frac{p_K}{\alpha A_K^p}\right)^{\frac{p}{1-p}} - \alpha A_K^p}{(1-\alpha)A_L^p} = \frac{\alpha A_K^p}{\left(\frac{p_L}{A_L^p(1-\alpha)}\right)^{\frac{p}{1-p}} - (1-\alpha)A_L^p}.$$

From here, after some transformations, the following equation of the price curve is received:

$$\alpha^{\frac{1}{1-p}}\left(A_K p_K\right)^{\frac{p}{p-1}} + (1-\alpha)^{\frac{1}{1-p}}\left(A_L p_L\right)^{\frac{p}{p-1}} = 1,$$

which can be written in form of the CES function:

$$B\left(\beta\left(\frac{p_K}{A_K}\right)^q + (1-\beta)\left(\frac{p_L}{A_L}\right)^q\right)^{1/q} = 1,$$

where $B = \left(\alpha^{\frac{1}{1-p}} + (1-\alpha)^{\frac{1}{1-p}}\right)^{\frac{p-1}{p}}$, $\beta = \dfrac{\alpha^{\frac{1}{1-p}}}{\alpha^{\frac{1}{1-p}} + (1-\alpha)^{\frac{1}{1-p}}}$, $q = \dfrac{p}{p-1}$.

Here $\tilde{p}_K = \dfrac{p_K}{A_K}$, $\tilde{p}_L = \dfrac{p_L}{A_L}$ are prices of the effective capital, $A_K K$, and the effective labor, $A_L L$; the expenditures can be calculated both using the production factors themselves or the effective factors:

$$p_K K = \tilde{p}_K \cdot A_K K, \;\; p_L L = \tilde{p}_L \cdot A_L L.$$

Notice that $q \in (-\infty, 0) \cup (0,1)$ and, in the same way as under the Cobb-Douglas production function, the price curve is concave (convex down). However under a low elasticity of substitution of the CES production function (to be precise, under $p \in (-\infty, 0)$ and, correspondingly, $q \in (0,1)$) the prices of factors are boarded, whereas under a high elasticity of substitution (under $p \in (0,1)$ and $q < 0$) an arbitrarily high price of one of the factors is possible.

4.2 Reversed causality

Usually it is supposed that the prices are primarily determined by the physical side of production – physical technologies and existing bundles of production resources. However, another direction of causality is possible: institutions reflected by the prices can define which products will be produced and by use of which technologies[6].

We propose now a model in which the price curve, Π, is defined in a pure institutional way. This model belongs to a class of island models – such where partially independent segments of a market are considered.

There are two types of agents: workers and entrepreneurs. A single product is produced in a continuum set of segments – *islands*; some of them are "inhabited" by the agents of both types. On each of the inhabited islands in each moment of time there are definite prices (payment rates) of labor and capital in terms of the product. In random moments of time from randomly chosen islands either a part of workers or a part of entrepreneurs moves to an uninhabited island. At this moment the prices in the inhabited island are fixed. After that a part of the other social group also moves from the "old" island to the "new" one and there the groups start bargaining about the factor prices. Those who have come first possess an advantage and try to increase their factor price – they *attack*. Those who have come later try not to allow their factor price to fall too much – they *defend*. As starting prices in the bargaining process the groups use the prices which they had had in the "old" island at the moment when the first group left. It is assumed that the social groups always have constant bargaining powers, b_K, b_L. Weakening this assumption is left for a future research.

Opposed to the case of the selling/purchasing bargaining game considered in Section 2, now the prices relate to different goods (labor and capital). The attacker, *a*, is interested in maximizing the growth rate of her factor price while the defender, *d*, is interested in minimizing (the module of) the growth rate of her factor price.

In the simplest case, similarly to the case considered in Section 2, it can be assumed that players have constant growth rates of their factor prices, $g_i = \frac{\dot{p}_i}{p_i}$; $g_a > 0$ for the attacker; $g_d < 0$ for the defender; and the price growth rates are linked with the bargaining powers by the equation:

$$|g_d| = \frac{b_a}{b_d} g_a . \quad (14)$$

According to this equation, a higher relative bargaining power b_d / b_a of the defender allows her to reach a slower decline in her factor price, i.e. a smaller $|g_d|$. Vice versa, an increase in the bargaining power of the attacker forces the defender to agree to a larger decline in her factor price.

Equation (14) describing the price change process turns into:

[6] This reminds a situation in a famous discussion about a relation between geographic factors and institutions (see e.g. Acemoglu et al., 2005): it is usually supposed that historically geographic conditions define technologies and the latter define institutions; but it is possible that vice versa institutions define a choice of places of settling as well as products to be produced and technologies to be used.

$$\frac{dp_a}{dp_d}\frac{p_d}{p_a} = -\frac{b_d}{b_a} = const \,, \quad (15)$$

which can be written as

$$\frac{dp_K}{p_K} b_K = -\frac{dp_L}{p_L} b_L$$

Solving this differential equation we receive

$$\ln {p_K}^{b_K} = -\ln {p_L}^{b_L} + const \,,$$

and, hence,

$${p_K}^{b_K} {p_L}^{b_L} = C \,. \quad (16)$$

Thus, the price curve Π is described. If initially the price vector belongs the curve Π given a constant *C* then the vector stays in the same curve further.

To describe the strategic behavior of the players in more details, let the attacker's problem be to maximize her price growth rate, g_a, under the following constraint:

$$|g_d| \geq g_a \frac{b_a}{b_d} \,, \quad (17)$$

and, correspondingly, let the defender's problem be to minimize the module of her price growth rate, $|g_d|$, under (17). The inequality (17) means that the attacker forces the defender to increase her price reduction rate. An increased bargaining power of the attacker reinforces this constraint while an increase in the bargaining power of the defender relaxes it.

In Figure 1 a solution of the maximizing player (attacker) under a fixed strategy of the defender is shown, and Figure 2 shows the solution of the defender under a fixed strategy of the attacker.

There exists a continuum of Nash equilibria, (g_a, g_d), and all of them satisfy the equation

$$\frac{g_a}{|g_d|} = \frac{b_d}{b_f} \,.$$

This equation, independently on which player (K or L) is the attacker, reduces to (15) and we come to the price curve (16).

Now let us show in what way the price curve (16) leads to the Cobb-Douglas type of production function.

We will use the representation of neoclassical production function by use of a menu of Leontief technologies (Matveenko, 1998, 2010, Jones, 2005). Matveenko (2010) has shown that to each neoclassical production function $F(K, L)$ a unique technological menu $\Psi = \{l = (l_K, l_L)\}$ corresponds which consists of effectiveness coefficients of the Leontief function and is such that

$$F(K, L) = \max_{l \in \Psi} \min\{l_K K, l_L L\} \,. \quad (18)$$

Figure 1: Solution of the attacker under a fixed strategy of the defender

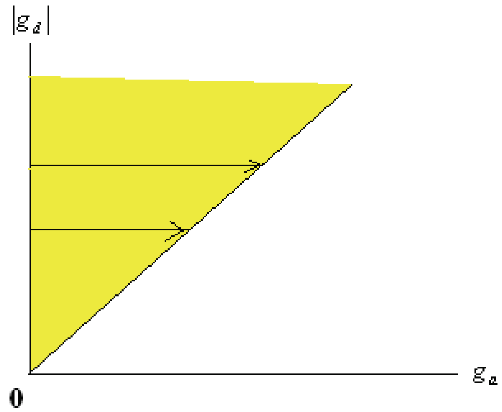
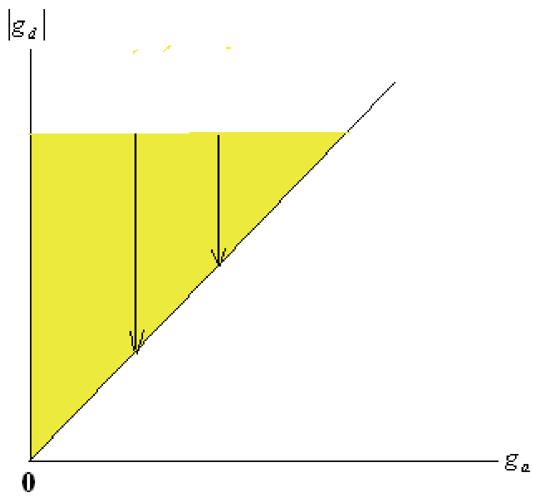


Figure 2: Solution of the defender under a fixed strategy of the attacker



Moreover, there exists a simple one-to-one correspondence between the points $(K, L) \in M$ of the factor curve and the points $l \in \Psi$ of the technological menu:

$$(l_K, l_L) \in \Psi \leftrightarrow \left(\frac{1}{l_K}, \frac{1}{l_L}\right) = (\tilde{K}, \tilde{L}) \in M .$$

The function

$$F^{\circ}(l_K, l_L) = \frac{1}{F\left(\frac{1}{l_K}, \frac{1}{l_L}\right)}$$

is referred to as a *conjugate function*. Representation (18) follows from the following Lemma .

LEMMA 1. *Let $F(x_1, x_2, \ldots, x_n)$ be an increasing positively homogeneous of 1st power (i.e. CRS) function of n positive variables, M – its unit level set, and Ψ – the unit level set of the conjugate function:*

$$M = \{x : F(x_1, x_2, \ldots, x_n) = 1\} ,$$

$$\Psi = \{l : F\left(\frac{1}{l_1}, \frac{1}{l_2}, ..., \frac{1}{l_n}\right) = 1\}.$$

Then

$$F(x_1, x_2, ..., x_n) = \max_{l \in \Psi} \min\{l_1 x_1, l_2 x_2, ..., l_n x_n\}. \quad (19)$$

Proof. For any $\tilde{x} \in (x_1, x_2, ..., x_n) \in M$ and any $l = (l_1, l_2, ..., l_n) \in \Psi$ equation

$$\min\{l_1 \tilde{x}_1, l_2 \tilde{x}_2, ..., l_n \tilde{x}_n\} = 1 \quad (20)$$

holds if $l = \left(\frac{1}{\tilde{x}_1}, \frac{1}{\tilde{x}_2}, ..., \frac{1}{\tilde{x}_n}\right)$, and inequality

$$\min\{l_1 \tilde{x}_1, l_2 \tilde{x}_2, ..., l_n \tilde{x}_n\} < 1$$

holds if $l \neq \left(\frac{1}{\tilde{x}_1}, \frac{1}{\tilde{x}_2}, ..., \frac{1}{\tilde{x}_n}\right)$. Hence,

$$F(\tilde{x}) = 1 = \max_{l \in \Psi} \min\{l_1 \tilde{x}_1, l_2 \tilde{x}_2, ..., l_n \tilde{x}_n\} \quad (21)$$

for each $\tilde{x} \in M$. Any vector $x = (x_1, x_2, ..., x_n)$ with positive components can be written as $x = \mu \tilde{x}$, where $\tilde{x} \in M$ (evidently, $\mu = F(x)$). By virtue of homogeneity, (21) implies (19) for each positive vector. Q.E.D.

When a pair of prices is defined on an island, the island chooses a suitable technology on base of one or another pure economic criterion (efficiency) or an institutional criterion (fairness). We assume that the whole set ("cloud") of available Leontief technologies is extensive enough to include all those technologies which any islands would choose to use. The technological menu Ψ is narrower and consists of those technologies which can be chosen.

Below three mechanisms of choice are identified resulting in the same technological menu Ψ and the factor curve *M* .

Mechanism A. Given factor prices $\left(p_K^0, p_L^0\right)$, an island chooses such Leontief technology (l_K, l_L) which guarantees receiving factor shares equal to the relative bargaining powers of the social groups[7]: $\alpha = \frac{b_K}{b_K + b_L}$ for the capital and $1 - \alpha = \frac{b_L}{b_K + b_L}$ for the labor.

For this technology, such volumes of factors $\tilde{K}, \tilde{L}$ for which:

$$l_K \tilde{K} = l_L \tilde{L} = 1,$$
$$p_K^0 \tilde{K} = \alpha, \; p_L^0 \tilde{L} = 1 - \alpha.$$

This choice of the Leontief technologies by the islands results in the following factor curve:

$$M = \{(K, L) : p_K K = \alpha, \; p_L L = 1 - \alpha, \; (p_K, p_L) \in \Pi\} =$$
$$= \left\{(K, L) = \left(\frac{\alpha}{p_K}, \frac{1 - \alpha}{p_L}\right), \; (p_K, p_L) \in \Pi\right\} =$$

[7] A support for this assumption is provided in Section 4.

$$=\left\{(K,L):\left(\frac{\alpha}{\mathrm{K}}\right)^{\alpha}\left(\frac{1-\alpha}{L}\right)^{1-\alpha}=C\right\},$$

i.e.

$$M=\{(K,L):\ AK^{\alpha}L^{1-\alpha}=1\},$$

where $A=\dfrac{C}{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}$.

Thus, the Leontief technologies chosen by the islands define the Cobb-Douglas production function:

$$F(K,L)=AK^{\alpha}L^{1-\alpha}.$$

Mechanism B. Given factor prices $\left(p_K^0, p_L^0\right)$, an island chooses such Leontief technology $(l_K, l_L)=\left(\dfrac{1}{K^0},\dfrac{1}{L^0}\right)$ which is *competitive* in the sense that, under this technology, the cost of the unit production on the island is equal to 1 while the cost on any other island is greater than 1. So, the usage of this technology is profitable only on the present island. In other words,

$$p_K^0 K^0 + p_L^0 L^0 = 1 < p_K K^0 + p_L L^0$$

for any bundle of prices $(p_K, p_L)\in\Pi$, $(p_K, p_L)\neq(p_K^0,\ p_L^0)$.

It follows that $(p_K^0,\ p_L^0)$ is a solution of the problem

$$\min_{(p_K,p_L)\in\Pi}\{p_K^0 K^0 + p_L^0 L^0\}.$$

The first order optimality condition for this problem is:

$$\frac{p_L^0}{p_K^0}=\frac{1-\alpha}{\alpha}\frac{K^0}{L^0},$$

hence the factor shares ratio is

$$\frac{p_L^0 L^0}{p_K^0 K^0}=\frac{1-\alpha}{\alpha},$$

and we come to the Mechanism A.

Mechanism C. Given factor prices $\left(p_K^0, p_L^0\right)$, an island chooses a Leontief technology $(l_K, l_L)\in\Psi$ (or, what is equivalent, $(K,\ L)\in M$) ensuring fulfillment of a fairness principle:

$$\max_{(K,L)\in M}\min\left\{\frac{p_K^0 K}{b_K},\ \frac{p_L^0 L}{b_L}\right\},$$

which is analogous to the *Rawlsian maximin principle*: a gain of the most hurt agent has to be maximized. Here the gain of an agent is her revenue but with account of her bargaining power: a participant's gain increases if her relative bargaining power increases.

The solution is characterized by the equation:

$$\frac{p_K^0 \widetilde{K}}{b_K} = \frac{p_L^0 \widetilde{L}}{b_L}.$$

Hence,

$$\frac{p_K^0 \widetilde{K}}{p_L^0 \widetilde{L}} = \frac{\alpha}{1-\alpha},$$

and again we come to the Mechanism A.

5. Differential game of weights formation

In this Section we provide a microfoundation for the Mechanism A decribed in Section 4. We propose a differential game in which the players (workers and capital-owners) form a weight curve – a set of possible assessments (weights); the curve is used by an arbiter to choose a vector of weights in a concrete bargain[8].

Three common features present in many real bargains and negotiations. Firstly, it is a presence of an arbiter in which role often a community acts, in a framework of which the bargainers interact. Examples are so called 'international community', including governments and elites of countries, and different international organizations; a 'collective' or a union in a firm; a local community; a "scientific community", etc. The community acts as an arbiter realizing a control for bargains in such way that unfair, from the point of view of the arbiter, bargains are less possible, at least as routine ones. An outcome of an unfair bargain can be, with a help of the arbiter, revised, if not formally than through a conflict. Such conflicts rather often arise, both on a local and on a national levels, as well as in international relations.

Secondly, bargains inside a fixed set of participants are often not 'one-shot' but represent a routine repeated process in which a 'public opinion' of the community is important; and the latter is being formed along with the bargains. Usually it is unknown in advance what concrete bargains will take place and in what time, and the process of formation of the public opinion processes uninterruptedly to prepare it for future bargains. The public opinion can be modeled as a set of the vectors of weights – the moral-ethical assessments which can be used by the arbiter as coefficients for the participants' utilities[9]. Possibilities of formation of public opinion are limited both by possibilities of access to media and by image-making abilities of the participants.

Thirdly, the moral-ethical assessments formed by participants are usually not univalent, but allow a variance: the public opinion practically always can stress both positive and negative features of a participant; concrete weights differ in different concrete bargains depending on obstacles. Thus, it is often useful to speak not about a single vector of weights but rather about a curve (in case of two participants) or a surface of admissible assessments.

Thus, the public opinion can be modeled as a weight curve (or a weight surface). In its approval or disapproval of a possible result of a concrete bargain the arbiter acts in accordance with a Rawlsian-type maximin principle, paying attention to the most infringed participant, but taking into account admissible vectors of weights for utilities; the set of admissible weights is formed in advance by the participants.

Let us consider a two stage game. On the first stage two players (workers and capital-owners) form a curve $\Lambda = \{(\lambda_K, \lambda_L)\}$ consisting of vectors of admissible reputational assessments (*weights*). On the second stage, for a concrete bargain, an arbiter (community) chooses an ad-

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[8] The weights of personal utilities are actively used in the bargaining theory (e.g. Shapley, 1969, Yaari, 1981) however there were almost no studies on the origin of the weights and on their relation to bargaining powers.

[9] Examples of reputational assessments of labor and capital are alternative opinions formed in the society on a special role of top-management in a modern production and on a decrease in the labor share as a result of globalization.

missible pair of weights from the weight curve Λ and divides the product *Y* among the players ($Y = Y_K + Y_L$) to achieve the maximin[10]

$$\max_{y \in \Omega} \max_{\lambda \in \Lambda} \min\{\lambda_K Y_K, \lambda_L Y_L\}. \quad (22)$$

Here

$$\Omega = \{y = (Y_K, Y_L\} : Y = Y_K + Y_L\}$$

is the set of outputs.

Let us describe the first stage of the game in detail. The *i*-th player's gain depends negatively on her weight λ_i and depends positively on the opponent's weight λ_j, $j \neq i$. Hence, each player i is interested in decreasing her weight, λ_i, and in increasing the opponent's weight λ_j. However, in the process of the weight curve formation, the player *i* would agree to a decrease in the opponent's weight in some part of Λ at the expense of an increase in her own weight, as far as the opponent similarly temporizes in another part of Λ.

Since the system of weights is essential only to within a multiplicative constant, the players can start the formation of the weight curve Λ from an arbitrary pair of weights and then construct parts of the curve Λ to the left and to the right of the initial point.

The player who *attacks* decreases her weight while the *defender* does not allow her weight to increase too much. At each moment of time the attacker maximizes the module $|g_a|$ of her weight's growth rate and the defender minimizes the growth rate of her weight g_d under the following constraint:

$$|g_a| \leq g_d \frac{b_a}{b_d}, \quad (23)$$

which means that a higher bargaining power of the attacker helps her to enlarge the constraint, while an increase in the bargaining power of the defender makes the constraint stricter[11].

In equilibrium (23) is fulfilled as an equality. Thus, the constancy of the bargaining powers of the participants implies:

$$\frac{d\lambda_L}{dl_K} \frac{\lambda_K}{\lambda_L} = -\frac{b_K}{b_L} = const. \quad (24)$$

It means that the workers agree to a 1% decrease in the entrepreneurs' weight just as they agree only to a $\frac{b_K}{b_L}$% increase in their own weight. The more the bargaining power of a player is the better the reputational assessment she gains for herself is.

Solving the differential equation (24) we receive the weight curve Λ:

$$\ln \lambda_L^{b_L} = -\ln \lambda_K^{b_K} + const,$$

which can be rewritten in the form:

$$\lambda_K^{b_K} \lambda_L^{b_L} = C = const.$$

[10] According to the common Rawlsian maximin criterion the most restrained participant has to receive maximum utility. Notice, that when using the weighted maximin criterion the arbiter "plays into a hand" of the wealthier player by increasing the weight of the restrained player.

[11] In the same way as in Section 3, it is assumed that only the inequality directly related to the actions of the most active player (the attacker) is important as a constraint.

5.1 Properties of the weight curves

Let us see how the position of the weight curve depends on the relative bargaining power. Let the players start formation of the weight curve from a point $(\hat{\lambda}_K, \hat{\lambda}_L)$. Then the equation of the weight curve is:

$$\lambda_K^{\ b_K} \lambda_L^{\ b_L} = C = \hat{\lambda}_K^{\ b_K} \hat{\lambda}_L^{\ b_L},$$

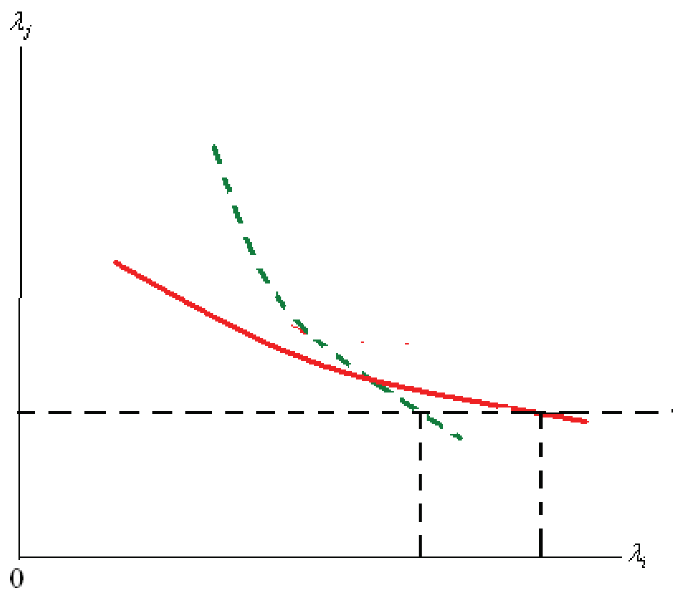
or, in an explicit form,

$$\lambda_i = \left(\frac{\hat{\lambda}_j}{\lambda_j} \right)^{\frac{b_j}{b_i}} \hat{\lambda}_i . \qquad (25)$$

Under $\lambda_j < \lambda_j$ player *j* attacks (i.e. diminishes her weight) and player *i* defends (prevents increasing her weight). In this situation an increase in the relative bargaining power of player *i* (i.e. a decrease in b_j / b_i) would provide, according to (25), a decrease in her weight λ_i when λ_j is fixed. In other words, the defender achieves the more success in defense (i.e. a lower weight) the higher her relative bargaining power is. This is illustrated in Fig. 3.

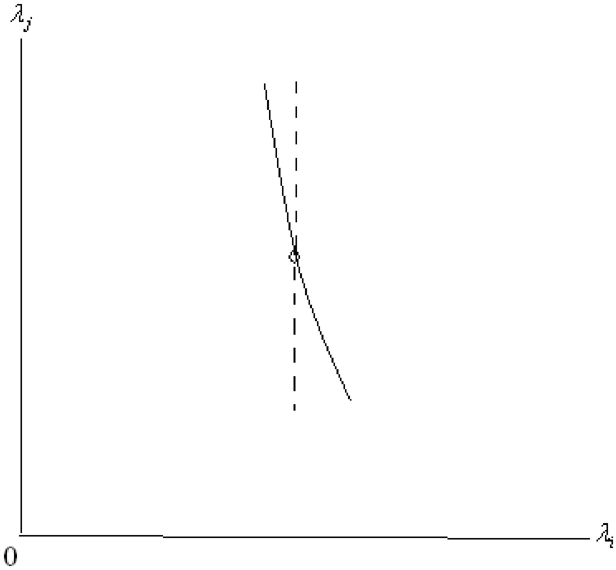
Evidently, the attacker also gains from her higher bargaining power, as far as the opponent's weight becomes higher.

Figure 3: Comparison of weight curves corresponding different relative bargaining powers. For the dashed (green) weight curve the relative bargaining power of player *i* is higher than for the solid (red) weight curve. When player *j* attacks (moves down), the dashed curve is preferable for the defender (player *i*); correspondingly, the solid curve is preferable for the attacker (player *j*).



If the relative bargaining power of player *i* goes to infinity, i.e. $\frac{b_j}{b_i} \to 0$, then, according to (25), $\lambda_i \to \hat{\lambda}_i$. It means that if player *j* attacks but player *i* possesses a very high bargaining power then the weight of player *i* increases only negligibly (see Fig. 4). But if player *i* attacks then, with an increase in $\hat{\lambda}_i / \lambda_i$, the weight of player *j* increases significantly.

FigURE 4: The weight curve under a very high relative bargaining power of player *i*.



5.2 Generation of the asymmetric Nash bargaining solution

Now we turn to the second stage of the game.

LEMMA 2 *For each outcome* $y \in \Omega$*, the following equality is valid:*

$$\max_{\lambda \in \Lambda} \min\{\lambda_K Y_K, \lambda_L Y_L\} = A Y_K^{\frac{b_K}{b_K+b_L}} Y_L^{\frac{b_L}{b_K+b_L}}, \quad (26)$$

where $A = const$.

Proof. Applying Lemma 1 to the set Λ we receive:

$$\max_{\lambda \in \Lambda} \min\{\lambda_K Y_K, \lambda_L Y_L\} = A Y_K^{\beta_K} Y_L^{\beta_L}$$

for any point (Y_K, Y_L); here $A = C^{-\frac{1}{b_K+b_L}}$. Q.E.D.

According to (26), the arbiter's problem (22) reduces to:

$$\max Y_K^{b_K} Y_L^{b_L} \quad (27)$$

s. t.

$$y \in \Omega. \quad (28)$$

The solution of this problem is none other than the *asymmetric Nash bargaining solution.*

For any outcome $y \in \Omega$ there exists a unique vector of weights $\overline{\lambda} \in \Lambda$, such that $\overline{\lambda}_K Y_{K1} = \overline{\lambda}_L Y_L$, namely,

$$\overline{\lambda}_i = C^{\frac{1}{b_K+b_L}} \left(\frac{Y_j}{Y_i} \right)^{\frac{b_i}{b_{1K} b_L}}, \; i, j = K, L, \; i \neq j.$$

The number

$$v(y) = \overline{\lambda}_1 Y_1 = \overline{\lambda}_2 Y_2$$

will be referred as a *utility of outcome*.

PROPOSITION 2. $v(y) = \max_{\lambda \in \Lambda} \min\{\lambda_K Y_K, \lambda_L Y_L\}$.

Proof. Let $\lambda \in \Lambda$ be an arbitrary vector of weights, $\lambda \neq \overline{\lambda}$. Then

$$\min\{\lambda_K Y_K, \lambda_L Y_L\} < \min\{\bar{\lambda}_K Y_K, \bar{\lambda}_L Y_L\} = v(y).$$

Q.E.D.

Thus, the arbiter's problem (22) is equivalent also to:

$$\max_{y\in\Omega} v(y).$$

5.3 Some other equivalent criteria

Similarly to Lemma 1 and Lemma 2 it can be proved that

$$\min_{\lambda\in\Lambda}\max\{\lambda_K Y_K, \lambda_L Y_L\} = AY_K^{\beta_K} Y_L^{\beta_L}.$$

It means that the "Pharisaical just" society (see footnote 9) for which the criterion is (22) does not differ, by its outcome, from a society searching for $\max_{y\in\Omega}\min_{\lambda\in\Lambda}\max\{\lambda_K Y_K, \lambda_L Y_L\}$ and, thus, openly acting in favor of the wealthier player by improving (i.e. decreasing) her weight[12].

The following proposition means that the same outcome can be received also as a *utilitarian solution*.

PROPOSITION 3. *Solution of the problem*

$$\max_{y\in\Omega}\min_{\lambda\in\Lambda}\left(b_K\lambda_K Y_K + b_L\lambda_L Y_L\right)$$

coincides with the solution of the problem (22).

Proof. For the sub-problem $\min_{\lambda\in\Lambda}\left(b_K\lambda_K Y_K + b_L\lambda_L Y_L\right)$ with a fixed *y*, maximization of the Lagrange function,

$$b_K\lambda_K K + b_L\lambda_L L - \mu\left(\lambda_K^{b_K}\lambda_L^{b_L} - C\right),$$

leads to the first order optimality conditions:

$$b_K Y_K - \mu b_K \lambda_K^{b_K-1}\lambda_L^{b_L} = 0,$$

$$b_L Y_L - \mu b_L \lambda_K^{b_K}\lambda_L^{b_L-1} = 0.$$

It follows that

$$\lambda_K Y_K = \lambda_L Y_L$$

The rest follows from Proposition 2. Q.E.D

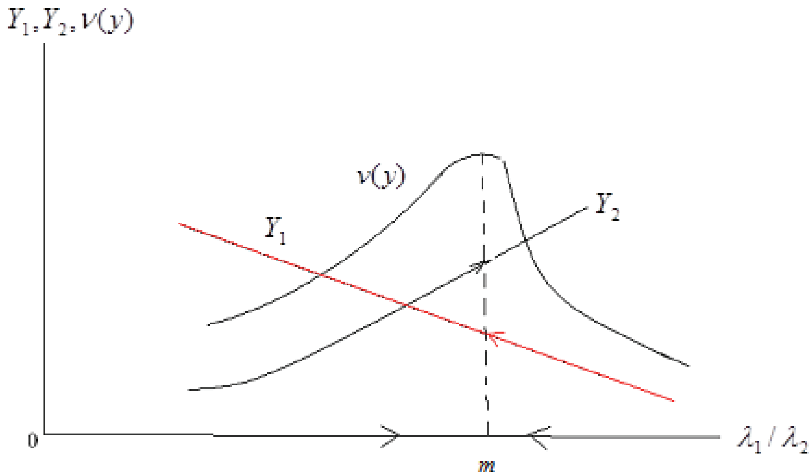
5.4 Moral-ethical assessments as a mechanism

Earlier we supposed that the weight curve Λ is constructed in advance and then the arbiter (community) uses it in any concrete bargain. Now let us consider another version of the model: in a concrete bargain the participants change weights under a control of the arbiter, and the latter does not allow the value of outcome to diminish.

[12] The model thus demonstrates that societies with different political mechanisms can have no considerable differences in their economic characteristics such as income distribution.

If at a current moment of time the value of outcome decreases in λ_i then player *i* can attack decreasing her weight, λ_i, and increasing the opponent's weight, λ_j. Player *j* in this situation can only defend, because her attack would decrease $v(y)$ what is not allowed by the arbiter. The attack of player *i* can continue only as long as the value of outcome, $v(y)$, increases. The dynamics of the weights is described by the differential game introduced above in this Section. The process stops as soon as the value of outcome reaches its maximum (Fig. 5).

Figure 5: Directions of changes under a control by the arbiter. The value of outcome achieves its maximum in point *m*. The arbiter allows the capital-owners to attack if $\lambda_1 > m\lambda_2$; the workers are allowed to attack if $\lambda_1 < m\lambda_2$.



5. 5 Generation of the Cobb-Douglas production function

Reduction to the Mechanism A. It is easily seen that in the solution of the problem (27)-(28) the players receive shares proportional to their bargaining powers: $\frac{Y_K}{Y_L} = \frac{b_K}{b_L}$. Proposition 2 implies $\frac{Y_K}{Y_L} = \frac{\lambda_L}{\lambda_K}$.

Hence, the pair of the weights (λ_K, λ_L) for which $\frac{\lambda_L}{\lambda_K} = \frac{b_K}{b_L}$ will be chosen by the arbiter (community). In such way, the social groups have to receive shares of the product proportional to their bargaining powers.

This provides a support to the Mechanism A described in Section 3. This mechanism, as we have seen, generates the Cobb-Douglas production function.

Notice, that a constancy of bargaining powers can explain a constancy of factor shares in some countries on a definite stage of their development – a validity of the corresponding Kaldor's stylized fact of economic growth.

Alternative way. There is a different way to explain the formation of the Cobb-Douglas production function in the context of weight curve. Let us assume that an economy consists of segments (islands) *i* combining inputs (labor and capital) in different proportions. For each bundle of inputs $(K(i), L(i))$ a pair of weights $(\lambda_K, \lambda_L) \in \Lambda$ is selected by the participants and by the arbiter in such way that a parity condition takes place:

$$\frac{K(i)}{\lambda_K(i)} = \frac{L(i)}{\lambda_L(i)}.$$

This can be interpreted as a demand of "equal" efforts of two participants, when a better reputation (a lower weight) allows a participant to include less efforts. At the same time, there is a demand for "equal" distribution of efforts for a unit production in the whole economy:

$$\frac{K(i)}{\lambda_K(i)} = \frac{L(i)}{\lambda_L(i)} = E = const. \quad (29)$$

To satisfy the institutional such a technology has to be chosen for which (29) is true. From the equation of the weight curve it follows that

$$\left(\frac{K(i)}{E}\right)^{b_K}\left(\frac{L(i)}{E}\right)^{b_l} = C.$$

In such way the unit production will be received under

$$AK^{\alpha}L^{1-\alpha} = 1,$$

where $A = \frac{1}{C^{\frac{1}{b_K+b_L}}E}$, $\alpha = \frac{b_K}{b_K+b_L}, 1-\alpha = \frac{b_L}{b_K+b_L}$. Thus we come to the Cobb-Douglas production function again.

6. Conclusion

In this paper a new approach is proposed for understanding a relation between a physical side of economy (resources and technologies) and its institutional side (distributional relations between social groups). The idea of the models presented here is that the distributional behavior can be described by a differential game of bargaining. A dual relation between the institutional and the physical sides of the economy allows to achieving an independent description of production function on base of a differential game in the institutional side. Thus, institutions can be a primal reason of a choice of technologies and, ultimately, define a production function.

Three differential games are proposed to describe a behavior of economic agents in processes of prices and weights formation. In the benchmark model of price bargaining players are interested in changing the same price in opposite directions. It is shown that under some conditions this game leads to the Nash bargaining solution. This benchmark game is modified to games in which players change (different) prices of their owned resources or change weights (moral-ethical assessments). One of these games describes bargaining of workers and capital-owners for their factor prices. In another game the same players bargain for weights (moral-ethical assessments); these weights enter a Rawlsian-type criterion which is used by an arbiter (community) in concrete bargains.

These games result in construction of structures – a price curve in one case and a weight curve in another – which are dual to the production function. Ultimately, under constant bargaining powers of the participants, these games lead to the Cobb-Douglas form of production function.

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