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Montenegrin Journal of Economics

Yerznkyan, B.H., Gataullin, T.M., Gataullin, S.T. (2021), "Solow Models with Linear Labor Function for Industry and Enterprise", *Montenegrin Journal of Economics*, Vol. 17, No. 1, pp. 111-120.

Solow Models with Linear Labor Function for Industry and Enterprise

BAGRAT H. YERZNKYAN[1], TIMUR M. GATAULLIN[2] and SERGEY T. GATAULLIN[3]

[1] Professor, Chief Researcher, Head of Lab. CEMI RAS, e-mail: lvova1955@mail.ru; yerz@cemi.rssi.ru
[2] Professor, Center for Digital Economy, State University of Management, e-mail: gataullin@inbox.ru
[3] Candidate of Sciences, Financial University under the Government of the Russian Federation, e-mail: gataullin097@gmail.com

ARTICLE INFO

Received April 28, 2020
Revised from May 29, 2020
Accepted July 30, 2020
Available online March 15, 2021

JEL classification: A10, B41, C00, O40

DOI: 10.14254/1800-5845/2021.17-1.8

Keywords:

Solow model,
labor resources' linear functions,
modification with Cobb-Douglas function and employees' linear increase,
provisions for industry and enterprise

ABSTRACT

The paper considers a modification of the Solow model with linearly changing labor resources. In the classical Solow model, the economy of the whole country is considered, the number of employed coincides with the entire able-bodied population and grows exponentially. The authors of the work thought that this is generally unrealistic when the number of employees grows exponentially and decided to study the case when the number of employees varies linearly. As far as the authors could trace, such a modification of the Solow model was not considered in the available literature. In addition, reasonable modifications of the Solow model for the industry and an individual enterprise were found in the work and all of these modifications were investigated for a special case of the Cobb-Douglas production function.

INTRODUCTION

The classical growth model (Solow, 1956; 1988) describes the economy of the whole country as a single sector where the growth in the number of people employed in it is exponential. Based on this model, economists and mathematicians have created a huge number of more complex modifications of it [see, for example: Tumanova and Chagas, 2004]. In almost all of these modifications, the growth in the number of employees is exponential (Snieska et al., 2020; Chen et al., 2019). However, instead of the assumption of an exponential increase in the volume of labor resources in the classical Solow model, the authors of this paper consider a modification with linearly changing labor resources. This, in particular, made it possible to find reasonable modifications of the classical model for the industry and an individual enterprise.

From economic point of view such assumption of linearly growth in the number of employees is more realistic. Note that, when the condition of the constant accumulation rate is rejected and the intermediate product is taken into account, the classical Solow model adjusted in this way becomes a dynamic problem of rational farming, which can be solved using the Pontryagin maximum principle [see, for example: Kolemaev, 2012). Moreover, the specific consumption c (t) is considered as a control parameter. The task of the governing body of the economic system is to manage the system in this way with the help of fiscal policy or directives, changing from (t) so that over a long period of time (conditionally accepted as infinity), the utility from consumption would be greatest or in accordance with the principle sufficiency (Gataullin, 2017; Kolemaev et al., 2008; Yerznkyan et al., 2018; Pisareva, 2017).

Great opportunities for developing ideas of the classical R. Solow model are associated with modeling in the knowledge economy (Machlup, 1966; Makarov and Kleiner, 2007; Formánek, 2019), in particular, using tropical mathematics and idempotent analysis (Maslov and Kolokoltsev, 1994; Gataullin, 1977; Gataullin and Karasev, 1971). In addition, the importance of formal and non-formal institutional environment for economic development (Yerznkyan and Gassner, 2018; Kostiukevych et al., 2020) with accent on social orders – limited access (North et al., 2007) and open access orders (North et al., 2009), not saying of economies in transition (Draskovic, 2018), is not taken into account. In this paper, without considering these and other less abstract and more realistic factors of economic growth, we will try to present some modified versions of the Solow model which as far as authors know was not considered in the available literature. These modifications present a special case of the Cobb-Douglas production function used for the industry and the enterprise.

The paper is organized as follows. Section 1 outlines the basics of the classical model of Solow. Section 2 presents a case when labor resources change linearly. Section 3 develops the Solow model for smaller structural units such as industry and enterprises. We conclude in the last section.

1. THE SOLOW GROWTH MODEL

The classical Solow model usually describes the economy of the whole country as a single sector where the growth in the number of people employed in it is exponential. Let us briefly recall this model in order to compare it with another particular case: when labor resources change linearly, as well as with the Solow model for smaller structural units: for the industry and for the enterprise. The state of the economy in the Solow model is defined by five variables: Y – final product of the entire economy (for example, in monetary terms), L – labor resources available, K – (production) funds (again in monetary terms), I – investments, C – size of non-productive consumption. All variables interconnected change over time, i.e. are functions of time *t*, but then the argument *t* will often be omitted, although it will be implied by default.

In the Solow model, time can be considered discrete, but the case of continuous time allows the use of a powerful apparatus of differential equations, so we will consider time as continuous. For instant indicators K, L, we can assume that K, L are funds and labor resources at time *t*, or, to avoid seasonal changes in the number of employees and a surge in funds when new capacities are commissioned, K and L can be considered average values of these values for the year, the middle of which is *t*. For the quantities Y, C, I, their value at time *t* can be imagined as their volumes accumulated over the year, the middle of which is the time *t* (but even in this case they remain functions of time and they are still better understood as production capacity and instant consumption and investment rates).

It is believed that resources (production and labor) are fully utilized. The annual final product at each moment of time is a function of the average annual funds and labor: Y = F (K, L), so F (K, L) is the production function of the entire economy. The final product is used for consumption and investment: Y = C + I. We call the rate of accumulation ρ the share of the final product used for investment, then: $I = \rho Y$, $C = (1-\rho)\,Y$. In the future, the rate of accumulation will be considered constant: $\rho = const, 0 < \rho < 1$.

Investments are used to restore retired funds and their growth. If we assume that the disposal of funds occurs with a constant coefficient of disposal of μ, $0 < \mu < 1$ (per year), then

$\Delta K = K(t + \Delta t) - K(t) = \rho Y \Delta t - \mu K \Delta t$, so $dK/dt = \rho Y - \mu K$.

In the classical Solow model, the increase in labor resources is proportional to the available labor resources: $\Delta L = vL\Delta t$, then we obtain the differential equation $dL / dt = vL$ and, solving it, we get $L = L_0 e^{vt}$, where $L_0 = L(0)$ are the labor resources at the beginning of the observation, at $t = 0$.

Thus, the classical Solow model is defined by a system of equations 1.

$C = (1 - \rho)Y$
$Y = F(K, L)$
$L = L_0 e^{vt}$
$dK/dt = \rho Y - \mu K, K(0) = K_0$ (1)

The function $F(K, L)$ satisfies the requirements for production functions and is considered a homogeneous function of the first degree, i.e.$F(\lambda K, \lambda L) = \lambda F(K, L)$. Using its homogeneity and denoting the average labor productivity $y = Y / L$, and the average capital productivity $k = K / L$, we obtain $y = Y / L = F(K, L) / L = F(K / L, 1) = F(k, 1)$ and, if we denote the last function $f(k)$, then we get $y = f(k)$.

We simply denote the average labor productivity y as the labor productivity, and the average capital productivity k as the capital productivity. Next, we find the derivative of k with respect to t:

$dk/dt = d(K/L)/dt = K'L - KL'/L^2 = K'/L - K(L'/L^2) = (\rho Y - \mu K)/L - Kv/L = \rho y - (\mu + v)K$ We finally get,

$dk/dt = \rho f(k) - (\mu + v)k, k(0) = k_0 = K_0/L_0$. (2)

Entire macro-behavior of the model (1) is determined by equation (2) and the dynamics of labor $L = L_0 e^{vt}$.

Equation (2) is an equation with separable variables and an initial condition; therefore, it has a unique solution; however, it is not possible to explicitly solve it because of the non-specific form of the function $f(k)$. Therefore, usually only some special solutions of this equation are investigated.
Consider a stationary trajectory, i.e. one on which the capital productivity k is constant and therefore equal to its initial value: $k(t) = const = k_0$. But since probably not every initial value can be such a constant value, we denote it by k^0. This value of the capital productivity is called stationary. Of course, on the stationary trajectory $dk / dt = 0$. Let's consider how behave on a stationary trajectory macro-indicators K, L, C, I, Y.

According to equation (2), if $dk / dt = 0$, then $\rho f(k) - (\mu + v) k = 0$, i.e. k^0 is a solution of the equation

$\rho f(k) - (\mu + v) k = 0$. (3)

It is easy to prove that this equation has a solution, and the only one. So, what are K, L, C, I, Y for this solution on a stationary trajectory?

Since $L(t) = L_0 e^{vt}$, and $k = K(t) / L(t)$, then $K(t) = k^0 L(t) = k^0 L_0 e^{vt}$; similarly, $Y(t) = f(k^0) L(t) = f(k^0) L_0 e^{vt}$.

Further, $C(t) = (1 - \rho) Y(t) = (1 - \rho) f(k^0) L_0 e^{vt}, I(t) = \rho f(k^0) L_0 e^{vt}$.

Put it all together:

$L(t) = L_0 e^{vt}$

$K(t) = k^0 L_0 e^{vt}$;
$Y(t) = f(k^0) L_0 e^{vt}$
$C(t) = (1-\rho) f(k^0) L_0 e^{vt}$
$I(t) = \rho f(k^0) L_0 e^{vt}$

We conclude: on a stationary trajectory, all the main macro-indicators grow exponentially, in proportion to labor resources. We concretize the described general case as applied to the Cobb-Douglas production function $F(K, L) = AK^{\alpha}L^{1-\alpha}$, $A > 0$, $0 < \alpha < 1$. Since in this case $f(k) = F(k, 1) = AK^{\alpha}$, then equation (2) takes the form $dk / dt = \rho AK^{\alpha} - (\mu + v)k, k(0) = k_0$.

This is an equation with separable variables and its solution is

$$k(t) = (k_0^{1-\alpha} + \rho A/(\mu + v)[e^{(1-\alpha)(\mu+v)t} - 1])^{1/(1-\alpha)} e^{-(\mu+v)t}.$$

It's clear that $\lim_{t\to\infty} k(t) = [\rho A/(\mu + v)]^{1/(1-\alpha)}$.

But in our case, the equation $\rho f(k) - (\mu + v)k = 0$ has the form $\rho Ak = (\mu + v)k$ and the stationary value of the capital productivity for the Cobb-Douglas function is $k^0 = [\rho A / (\mu + v)]^{1/(1-\alpha)}$. Therefore, for any initial value of k_0, the capital productivity $k(t)$ converges to a stationary value k^0.

Since $y(t) = Ak^{\alpha}$, the labor productivity also converges to the stationary value $y^0 = A[\rho A / (\mu + v)]^{\alpha/(1-\alpha)}$. Therefore, the specific consumption (per worker) also converges to a stationary value:

$$\lim_{t\to\infty} C(t)/L(t) = \lim_{t\to\infty}(1-\rho)y(t) = (1-\rho)A[\rho A/(\mu + v)]^{\alpha/(1-\alpha)}.$$

In the study of the model, it is quite reasonable to take the value of specific consumption as a criterion for the success of economic development. We will find at what value of the rate of accumulation the maximum specific consumption equal to, as we saw, the maximum specific consumption in the stationary mode. To do this, we differentiate this specific consumption value $(1-\rho)A[\rho A/(\mu + v)]^{\alpha/(1-\alpha)}$ with respect to ρ and equate the derivative 0:$((1-\rho)A[\rho A/(\mu + v)]^{\alpha/(1-\alpha)})'_{\rho} = 0$,
$((1-\rho)\rho^{\alpha/(1-\alpha)})'_{\rho} = 0$, and after simple calculations we get
$\rho' = \alpha$.

So, the optimal rate of accumulation in stationary mode is equal to the coefficient of elasticity for funds (the "golden rule" of economic growth). But this is true for the Cobb-Douglas production function. For other production functions, this rule, quite possibly, will be different.

2. LINEAR FUNCTIONS OF LABOR RESOURCES

The assumption of an exponential increase in the volume of labor resources in the classical Solow model can be true only for a certain limited period of time – the population of the whole country cannot grow exponentially for a long time! Such growth is possible only at some initial transitional stage of development. Further, such growth should slow down and give way, for example, to linear, or even become generally zero, i.e. the number of people employed in the economy will become constant, or may begin to decrease. So $L(t) = L(0)(l + vt)$. Almost repeating the calculations leading to equation (2), we obtain an analog of this equation:

$dk / dt = \rho f(k) - (\mu + v / (1 + vt))k, k(0) = k_0 = K_0 / L_0$. (3)

Unlike (2), this equation is not an equation with separable variables, and it is not possible to solve it. Therefore, we examine only some of its special solutions. Are there any decisions on which the capital per unit of labor of k is constant, i.e. what in the classical model was called "stationary trajectories"? On such trajectories, $dk\ /\ dt\ =\ 0$; therefore, $\rho f\ (k)\ -\ (\mu\ +\ v\ /\ (1\ +\ vt))\ k\ =\ 0$, but since $f\ (k)$ is a function, $f\ (k)$ is also constant, which means $(\mu\ +\ v\ /\ (1\ +\ vt))$ is also a constant, which is possible only if $v\ =\ 0$, i.e. with a constant number of employees. When $v\ =\ 0$, equation (3) takes the form: $dk\ /\ dt\ =\ \rho f\ (k)\ -\ \mu\, k, k\ (0)\ =\ k_0\ =\ K_0 /\ L_0$ (4) and on a stationary trajectory $\rho f\ (k)\ =\ \mu\, k_0$, i.e. $f\ (k)\ =\ k_0 \mu / \rho$. So, with a constant number of employees, the stationary trajectory is completely determined by its initial value of the capital per unit of labor and labor productivity is also constant on it. Now let $v \neq 0$, for $v >\ 0$ the number of employees increases, for $v\ < 0$ it decreases. As follows from the previous subsection, there are no stationary trajectories in this case. Let us prove some properties of the solutions of equation (3).

First, consider the case of population growth, i.e. when $v >\ 0$.

Proposition 1. If $dk\ /\ dt\ (t_0) \geq\ 0$ for some t_0, then $dk\ /\ dt\ (t) >\ 0$ for any $t > t_0$.

The proof follows from the fact that $f\ (k)$ increases with increasing k, and the subtracted on the right-hand side of equation (3) decreases with increasing t. So, if at some point the capital per unit of labor, at least, ceases to decrease, then it will continue to increase. In particular, the following corollary from proposition 1 holds.

Corollary 1. If $f\ (k_0)\ -\ (\mu\ +\ v)\ k_0 \geq\ 0$, then the capital per unit of labor on this trajectory increases. Let us now see whether the capital per unit of labor can decrease all the time.

Proposition 2. If $\rho f\ (k)\ =\ (\mu\ /\ \rho)\ k_0$, then k decreases with increasing t, more precisely, $k\ =\ k_0 /\ (1\ +\ vt)$.

Indeed, equation (7) in this case has the form $dk\ /\ dt\ =\ k\ (-v\ /\ (l\ +\ vt))$, its solution is $k\ (t)\ =\ C\ /\ (l\ +\ vt)$ and the definition of the integration constant gives $C\ =\ k_0$.

Proposition 3. With a linear relationship labor productivity from the capital per unit of labor$y\ =\ sk$, the value $s_0\ =\ \mu\ /\ \rho$ is critical: for $s > s_0$the capital per unit of labor increases, and for $s \leq\ s_0$ it decreases for $t \geq\ 0$. Indeed, the second part of the sentence actually follows from proposition 2, and the first part - from proposition 1: because if $f\ (k) >\ (\mu\ /\ \rho)\ k$, then for some t_0 there will be $\rho f\ (k) >\ (\mu\ +\ v\ /\ (\ 1\ +\ vt_0))\ k$.

Proposition 4. If $dk\ /\ dt\ (t)\ < 0$ for any $t \geq\ 0$, then $k \longrightarrow 0$ as $t \longrightarrow \infty$.
Proof. Since always $dk\ /\ dt\ < 0$, it follows that $\rho f\ (k)\ < \mu\, k\ +\ kv\ /\ (1\ +\ vt) \geq\ 0$; therefore, $f\ (k)\ <=\ (\mu / \rho)\ k$. Therefore, $dk\ /\ dt\ < -kv\ /\ (1\ +\ vt)$ and therefore $k\ (t)$ is less than a function that is a solution of the equation $dz\ /\ dt\ =\ -z\ (v\ /\ (1\ +\ vt)$ under the same initial condition. But the solution to the latter the equations are $z\ (t)\ =\ z_0\ /\ (1\ +\ vt)$. Thus, $k\ (t)\ < k_0\ /\ (1\ +\ vt)$ and since $k\ (t) \geq 0$, then $k\ (t)\ \longrightarrow 0$ for $t\ \longrightarrow \infty$.

It turns out the following interesting conclusion.

Proposition 5 . With a linear increase in the number of people employed on any trajectory, the capital per unit of labor either increases, starting at some point, or decreases all the time, striving for 0. Now consider the case of a decrease in the number of people employed in the industry, i.e. when $v\ < 0$. We assume that the parameter v is positive, but we rewrite equation (3) as follows
$dk\ /\ dt\ =\ \rho f\ (k)\ -\ (\mu\ - v\ /\ (1 - vt))\ k, k\ (0)\ =\ k_0\ =\ K_0 / L_0$. (5)

First of all, we note that the number of people employed in this industry cannot be negative, so the length of the period of linear decrease in number cannot be more than $T\ =\ 1\ /\ v$.

Proposition 6. Starting with a certain $t_0$the capital per unit of labor is increasing. Proof. Let's see when subtracted in equation (4) becomes negative (more precisely, non-positive), then $dk\ /\ dt$becomes positive. We have $\mu\ - v\ /\ (1 - vt)\ \leq 0$, and this the inequality holds for $t > t_0 = 1\ /\ v - 1\ /\ \mu$. It can be seen that $t_0 < T\ =\ 1\ /\ v$, so that the moment of the beginning of the increase in the capital per unit of labor will necessarily come. Can the capital per unit of labor increase all the time?

Proposition 7. On the trajectory, the capital per unit of labor increases all the time if and if only $f\ (k) \geq\ (\mu\ - v)\ k\ /\ \rho$.

Indeed, we have

$dk\ /\ dt > \ \rho\ f\ (k)\ -\ \mu k\ +\ vk\ +\ (kv\ /\ (1 - vt)\ - kv)\ =\ [\rho f\ (k)\ -\ (\mu\ - v)\ k]\ +$ $[kv\ /\ (1 - vt\ )\ - kv]$

The second square bracket at $t\ =\ 0$ is 0, so if the condition of the sentence is not fulfilled, then due to the negativity of the first square bracket, $dk\ /\ dt\ (0)\ < 0$. For $t >\ 0$, the second square bracket is positive, and the first square bracket is non-negative by the condition of the sentence, and therefore in this case $dk\ /\ dt >\ 0$. Let's now consider the Solow model with the Cobb-Douglas function and with a linear increase in the number of employees. For the Cobb-Douglas function $F\ (K, L)\ =\ A\ K^{\alpha} L^{1-\alpha}, 0 < \alpha < 1$, we have $f\ (k)\ =\ A\ k^{\alpha}$. Equation (3) in this case takes the form $dk\ /\ dt\ =\ \rho A k^{\alpha}\ -\ (\mu\ +\ v\ /\ (1\ +\ vt))\ k, k\ (0)\ =\ k_0\ =\ K_0 /\ L_0$. (5)

This is the Bernoulli equation and by changing the variable $k\ =\ z^{1/(1-\alpha)}$ it reduces to a linear $dz\ /\ dt\ =\ - (1 - \alpha)\ (\mu\ +\ v\ /\ (1\ +\ vt))\ z\ +\ (1 - \alpha)\ \rho A$. (6) As it is known, the general solution of such an equation is the sum of the general solution of the homogeneous equation $dz\ /\ dt\ =\ - (1 - \alpha)\ (\mu\ +\ v\ /\ (1\ +\ vt))\ z$ and some particular solution of the inhomogeneous equation (6). Omitting the calculations, we find the general solution of the homogeneous equation $z\ (t)\ =\ C\ (1\ +\ vt)^{-\ (1-\alpha)} e^{-\ (1-\alpha)\ vt}$.

Let $v >\ 0$.

Proposition 8. There is a particular solution $z_0(t)$ of equation (6) for which $\lim_{t\to\infty} z_0(t)\ =\ \rho A\ /\ \mu$. Proof. We write equation (6) in the form

$dz\ /\ dt\ =\ (1 - \alpha)\ \rho A -\ (1 - \alpha)\ (\mu\ +\ v\ /\ (1\ +\ vt))\ z,$

and since $v >\ 0$, it is clear that the desired particular solution $z_0(t)$ is the constant $\rho A\ /\ \mu$ with a small additive. So, the general solution of equation (6) is

$z\ (t)\ =\ \ C\ (1\ +\ vt)^{-\ (1-\alpha)} e^{\ -\ (1-\alpha)\ vt}\ \ +\ z_0\ (t)$. (7)

Proposition 9. For any solution $z\ (t)$ of equation (7)

$\lim_{t\to\infty} z\ (t) =\ \ \rho A\ /\ \mu$. Proof. For any integration constant C as $t\ \to \infty$, the first term in (7) tends to 0, and the second, as follows from proposition 8, to $\rho A\ /\ \mu$, and so $z\ (t)$ tends to $\rho A\ /\ \mu$.

However, all this can be proved in the following way: if $z\ (t) >\ \rho A\ /\ \mu\ +\ \varepsilon$, $\varepsilon >\ 0$, then such an inequality cannot hold for a long time, because for such $z\ (t)$, starting from some $t_0$inequality $z\ '(t)\ \ < (t - \alpha)\ \varepsilon$; and vice versa, if $z\ (t)\ \ < \rho A\ /\ \mu\ -\ \varepsilon, \varepsilon >\ 0$, then such an inequality also cannot hold for a long time, because for such $z\ (t)$, starting from some $t_1$inequality $z\ '(t)\ \ < (t - \alpha)\ \varepsilon$.

Proposition 10. In the Solow model with the Cobb-Douglas production function and with a linear increase in the number of employees on any trajectory, the capital per unit of labor tends to $(\rho A\ /\ \mu)^{1\ /\ (1-\alpha)}$, productivity tends to $A\ (\rho A\ /\ \mu)^{\alpha\ /\ (1\ - \alpha)}$, specific consumption tends to $(1 -\ \ \rho)\ A\ (\rho A\ /\ \mu)^{\alpha/(1-\alpha)}$. Indeed, returning to the variable k, we find that on any trajectory k tends to $(\rho A /\ \mu)^{1\ /\ (1-\alpha)}$. Given that labor productivity is equal to $y\ =\ Ak^{\alpha}$, we find that y tends to

$A\ (\rho A\ /\ \mu)^{\alpha/\,(1-\alpha)}$. Further, specific consumption, i.e. the consumption per worker is equal to $c\ (t)\ =\ (l-\rho)\,Y\ (t)\ /\ L\ (t)\ =\ (1-\rho)\ y\ (t)$ and tends to $(1-\rho)\ A\ (\rho A\ /\ \mu)^{\alpha/\,(1-\alpha)}$.

As in the classical model, we find for which p the specific consumption tends to the maximum value. Differentiating $(1-\rho)\ A\ (\rho A\ /\ \mu)\ \alpha/\ (1-\alpha)$ with respect to ρ, we get ρ* = a – exactly as in the classical model. Comparing with the corresponding provisions of the classical Solow model, we see that in a model with linear growth v seems to be reset to zero.

Solow model with Cobb-Douglas function and with linear decrease in the number of employees. We assume that v is the pace decrease, then $v>\ 0$. Equation (6) in this case takes the form
$dz\ /\ dt\ =\ (1-\alpha)\ \rho A-\ (1-\alpha)\ \mu z\ +\ (1-\alpha)\ vz\ /\ (1-vt)$. (8)

It can be seen that as the number of employees approaches 0, i.e. at $t\rightarrow 1\ /\ v$, $z\ (t)$ increases unboundedly, thereby the unlimited capital per unit of labor, labor productivity and specific consumption. However, this case is clearly unrealistic, and we will not investigate it further.

3. THE KEY POINTS OF SOLOW MODEL FOR INDUSTRY AND ENTERPRISE

The development of a sufficiently large and ramified industry, for example, the country's railways, can be described by a model based on the assumptions of the Solow model. Such a model will be called the Solow industry model. The industry's economy is considered as a whole (without structural units), goods, services, etc. produced by this industry. are valued in cash. The total score is how much the industry gives to the entire economy of the country and society. Naturally, the company must reimburse this amount of the industry (reflecting this amount, say, in the budget of the country), and the industry can spend the reimbursed amount on investments and the salary of its employees (what is called non-production consumption in the classical model).

The state of the industry in the Solow industry model is defined by the same five variables as above: Y is the end product of the industry (for example, in monetary terms), L is the available labor resources of the industry (for example, the number of employees in it), K is (production) sector funds (again in monetary terms), I – investment, C – size of non-productive consumption = total salary of industry workers.

It has already been noted that the assumption of an exponential increase in the volume of labor resources in the entire economy is clearly unrealistic. However, for the industry, such growth may well be, and not only at some initial stage of its development. In the end, of course, such growth should slow down and give way, for example, to linear, or even become zero at all, i.e. the number of people employed in the industry will become constant, or may begin to decrease. You can also notice that the population, the amount of available labor resources of the whole society is determined, generally speaking, not by economic categories, in contrast to the industry.

The industry model can also explore other issues. For example, a company can plan linear growth in industry products. The question is how should the number of people employed in the industry grow for this? Or how will the industry develop with a constant investment in it, at a constant salary of its employees, etc.? Some of these issues will be explored further. The Solow model can also be built for the enterprise. At the same time, questions arise that are qualitatively different than when studying the classical Solow model or the industry model of Solow. For example, how to reflect taxation issues in the model? But what remains of the Solow model? The fundamental remains: the production function $F\ (K,L)$ is considered known and satisfies the same conditions as in the classical Solow model – see above. The change in funds also satisfies the same conditions as above – investments are used to restore retired funds and their growth. So the amount of funds satisfies the equation

$dK\ /\ dt\ =\ \rho Y\ -\ \mu K, K\ (0)\ =\ K_0$. (9)

The volume of production of the industry (enterprise) is growing linearly. Such behavior, for example, for an industry can be set by society in the person of the state and its governing and planning bodies. So, $Y = Y_0 (1 + wt)$. If this is given, then how will the number of people employed in the shit change? It is important to note the following: if a change in the volume of industry products over time is specified, then a change in funds over time is thereby completely determined.
Indeed, if $Y(t) = \varphi(K, t)$, then according to equation (9),

$dK / dt = \rho\varphi(K, t) - \mu K, K(0) = K_0.$

Such an equation in many special cases can be completely solved. Let us return to the declared case of linear growth in industry volume. Equation (6) in this case has the form
$dK / dt = \rho Y_0 (1 + wt) - \mu K, K(0) = K_0.$ (10)

The resulting equation is inhomogeneous linear and for any initial condition has a unique solution. We find it by the method of variation of the constant. Common decision of the homogeneous equation $dK/dt = -\mu K, K(0) = K_0$ is $K(t) = Ce^{-\mu t}$.

Differentiating it and considering C as a function of t, substituting then in (10) we obtain $K(t) = Ce^{-\mu t} + \rho Y_0 / t + \rho Y_0 w (t/\mu - 1/\mu^2)$, we see that $K(t)$ for large t is $\rho t\, Y_0 w / \mu$.
Let the production function $F(K, L)$ be the Cobb-Douglas function $Y = A K^{\alpha} L^{1-\alpha}$, then we have $A K^{\alpha} L^{1-\alpha} = Y_0 (1 + wt)$, therefore, $L = [Y_0(1 + wt) / (A (K(t)^{\alpha}))]^{1/(1-\alpha)}$ and for large t $L(t)$ is $Y_0 wt (\rho A / \mu)^{-\alpha/(1-\alpha)}$.

Therefore, for large t, the return on assets is $Y(t) / K(t) = Y_0 wt (t Y_0 \rho w / \mu) = \mu / \rho$, labor productivity is $y = Y(t) / L(t) = (\rho A / \mu)^{\alpha/(1-\alpha)}$, the capital per unit of labor is $k(t) = K(t) / L(t) = \rho(A\rho / \mu)^{\alpha/(1-\alpha)}$, specific consumption is $c(t) = (1-\rho) Y(t) / L(t) = (1-\rho)(\rho A / \mu)^{\alpha/(1-\alpha)}$. It is very interesting that the ρ value, which maximizes the specific consumption, is the same as in the classical model and in the model with a linear population growth: $p* = \alpha$ (note that in these three modifications of the Solow model, including the Solow model itself, the production function there is a Cobb-Douglas function). Value added tax in the Solow model. Since revolving funds are not taken into account in the Solow model, $Y = F(K, L)$ gives the value of the added value of the enterprise, industry, GDP of the country for the studied time period. Based on this, elements of taxation, in particular VAT, can be added to the Solow model. At the same time, questions of reducing the tax burden can be considered as control.

CONCLUSION

65 years ago in February 1956 R. Solow published his classic basic model of exogenous economic growth in *The Quarterly Journal of Economics*. On Thursday, December 10, 1987, he was awarded the Nobel Prize in Economics at the Stockholm City Hall for his contribution to the theory of economic growth. In this model, as well as in many of its modifications, the growth in the number of employees is exponential. In this paper, we proposed a modified version of the Solow growth model with linearly changing resources. The main and other modifications were studied for a special case of the Cobb-Douglas production function. Contrary to the original version our model deals with meso and micro economic entities, such as industry and enterprises, and instead consideration exponentially growth of the number of employees supposes that their number varies linearly. It is worth to underline that in the case of the taking into account the intermediate product the original Solow model becomes a dynamic problem, which can be solved by the use of the Pontryagin maximum principle. Moreover, the specific consumption c (t) is considered as a control parameter.

It should be noted that in the case where the number of employees varies linearly, equation 1 for the capital per unit of labor of the classical model is replaced by equation 3, which is not an equation with separable variables and it is not possible to solve it. Therefore, as in the classical model, only some of its special solutions are investigated in the work, in particular, as in the classical model, stationary trajectories on which the capital per unit of labor is constant.

Three modifications of the R. Solow model considered in the work give new and interesting results for economists. For example, in the case of the classical model with a linear function of labor resources, the following unexpected result was obtained in Proposition 5: with a linear increase in the number of employees on any trajectory, the capital per unit of labor either increases, starting from some point, or decreases all the time, tending to zero. In the case of a linear decrease in the number of employees in proposal 6, a moment is found starting from which the capital per unit of labor increases. The answer to the question: can the capital per unit of labor increase all the time is obtained in Proposition 7, in which the necessary and sufficient condition for this is obtained. When studying this modification of the Solow model for a special case of the Cobb-Douglas production function, in Proposition 10 we find such an accumulation rate at which specific consumption tends to the maximum value. The optimal rate of accumulation turned out to be equal to the coefficient of elasticity for funds (the "golden rule of economic growth") - exactly as in the classical model. Modification of the classical model for the industry allows us to study many important questions for economists. For example, a company can plan a linear growth of industry products. The question is, how should the number of people employed in the industry grow for this? For the Cobb-Douglas production function, the answer is obtained in the penultimate paragraph of the third paragraph.

By modifying the classical model for an enterprise, one can answer qualitatively different questions than when studying the classical or industry model of Solow. For example, how to reflect taxation issues in the model? Since working capital is not taken into account in the Solow model, the final product gives the value of the added value of the enterprise, as well as the country's industry and GDP for the studied time period. Based on this, elements of taxation, in particular VAT, can be added to the Solow model. At the same time, questions of reducing the tax burden can be considered as controlling influences in the formation of the development trajectory.

It seems to us very interesting, in particular, for economists, that in the case of the Cobb-Douglas production function, the “golden rule of economic growth” remains true for all three modifications of the classical Solow model.

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